Recap:

pivotal quantities $Q(\underline{x}, \theta)$ whose distribution is free of the parameter.

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, μ, σ^2 both unknown.

Goal: Find 95% CI for σ^2.

$$\boxed{\frac{(n-1)S^2}{\sigma^2}} \sim \chi^2_{n-1}$$

pivotal quantity. This can be used to find 95% CI for σ^2

$$P\left(\chi^2_{n-1,0.025} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1,0.975}\right) = 0.95$$

$$\Leftrightarrow P\left(\frac{(n-1)S^2}{\chi^2_{n-1,0.975}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,0.025}}\right) = 0.95$$

95% CI for σ^2 is $\left[\frac{(n-1)S^2}{\chi^2_{n-1,0.975}}, \frac{(n-1)S^2}{\chi^2_{n-1,0.025}}\right]$.

Poisson Interval estimator:

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Pois}(\lambda)$, need to find $100(1-\alpha)\%$ CI for λ.

Note that $Y = \sum_{i=1}^{n} X_i \sim \text{Pois}(n\lambda)$. Let,

$$S = \left\{\lambda : F_\lambda(y) \leq 1 - \alpha/2,\ F_\lambda(y) \geq \frac{\alpha}{2}\right\}$$

where $F_\lambda(y)$ is the CDF evaluated at y of $\sum_{i=1}^{n} X_i$. Then S is a $100(1-\alpha)\%$ confidence set for λ.

$$F_\lambda(y) = \sum_{k=0}^{y} e^{-n\lambda} \frac{(n\lambda)^k}{k!} = P\left(\chi^2_{2(y+1)} > 2n\lambda\right)$$

This can be proved by using the fact that $\chi^2_{2(y+1)} \overset{\mathcal{L}}{=} \text{Gamma}\left(y+1, \frac{1}{2}\right)$ and by using integration by parts.

$$P\left(\chi^2_{2(y+1)} > 2n\lambda\right) = F_\lambda(y) \geq \frac{\alpha}{2}$$

this tells us that $\lambda < \dfrac{\chi^2_{2(y+1),\, 1-\alpha/2}}{2n}$

Also use the other equation

$F_\lambda(y) \leq 1 - \alpha/2$ to ensure that

$$\lambda > \frac{\chi^2_{2(y+1),\, \alpha/2}}{2n}$$

$$\Leftrightarrow S = \left\{\lambda : \frac{\chi^2_{2(y+1),\, \alpha/2}}{2n} < \lambda < \frac{\chi^2_{2(y+1),\, 1-\alpha/2}}{2n}\right\}$$

<u>Bayesian Credible intervals;</u>

In frequentist θ is a fixed but unknown quantity. Confidence interval is a random quantity as all X_i's are random.

- <u>Ex:</u> ~~$X_1, \dots, X_n$~~ $X_1, X_2 \overset{iid}{\sim} U\left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$

Since $X_i - \theta \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$, this is a location family.

$\bar{X} - \theta$ is a pivotal quantity.

$$P(-c \leq \bar{X} - \theta \leq c) = 1 - \alpha \quad \text{- - - } (1)$$

where, c is chosen from the distribution of $\bar{X} - \theta$. Note that, this distribution is free of θ.

① is equivalent to

$$P(\bar{X} - c \le \theta \le \bar{X} + c) = 1 - \alpha \quad \cdots ②$$

$\Rightarrow$ $100(1-\alpha)\%$ CI for θ is $[\bar{X}-c, \bar{X}+c]$.

Suppose you observe $X_1 = 1$, $X_2 = 2$. Then θ has to equal to 1.5.

$\Rightarrow$ $[\bar{x}-c, \bar{x}+c] = [1.5-c, 1.5+c]$ must contain θ.

As a Bayesian ~~we that~~ we assume the parameter θ is random and calculate posterior distribution of $\theta | X_1, \ldots, X_n$ from the prior distribution $\pi(\theta)$.

$$\pi(\theta | X_1, \ldots, X_n) \propto \pi(\theta) \prod_{i=1}^{n} f(x_i | \theta).$$

We define credible intervals of θ.

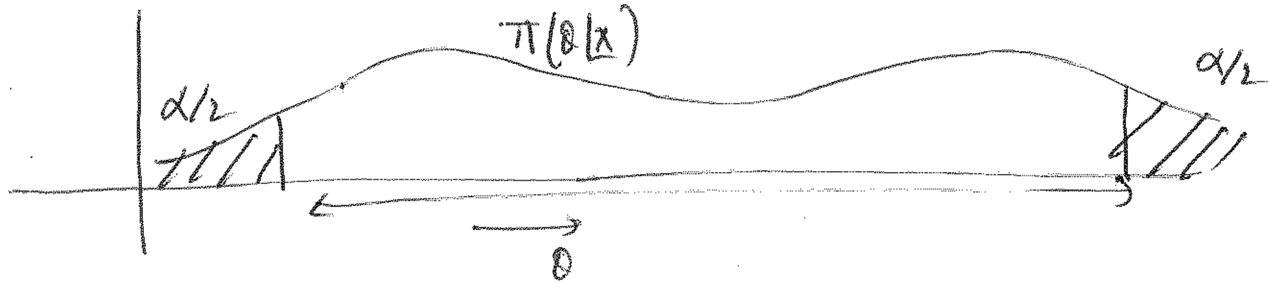
<u>Definition</u>: For $0 < \alpha < 1$, a $100(1-\alpha)\%$ credible set C is given by a set s.t. $P(\theta \in C \mid \underline{X}) = 1 - \alpha$.

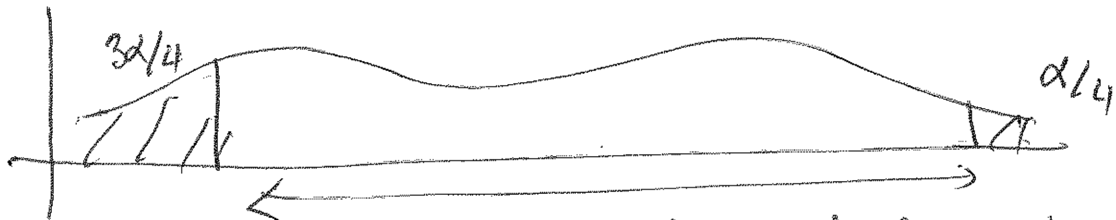
If the credible set is an interval, then it is called a credible interval.

If you have a posterior dist. $\pi(\theta | \underline{x})$, you are interested in finding an interval (a, b) s.t.

$$\int_a^b \pi(\theta | \underline{x}) \, d\theta = 1 - \alpha.$$

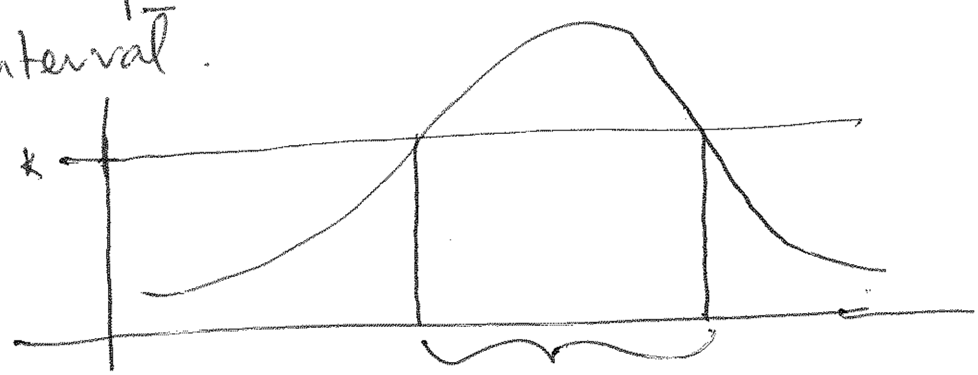
There are many different ~~ways to find~~ ~~ao~~ $100(1-\alpha)\%$ credible intervals.





Bayesians prefer to construct highest posterior density credible intervals.

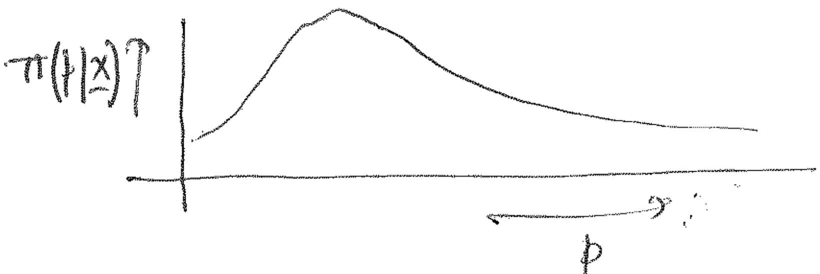
For a posterior density $\pi(\theta|\underline{x})$, the highest posterior density credible set is given by $C = \{\theta : \pi(\theta|\underline{x}) \geq k\}$. When ~~θ is unim~~ the $\theta|\underline{x}$ is unimodal C is actually an interval.



This is called the ~~HPDI~~ highest posterior density (HPD) interval.

Example: $X_1, \dots, X_n \overset{iid}{\sim} Ber(p)$, $p \sim Beta(a,b)$

$$p | X_1, \dots, X_n \sim Beta\left(a + \sum_{i=1}^n X_i \,,\, b + n - \sum_{i=1}^n X_i\right)$$



One can find HPD $100(1-\alpha)\%$ interval easily.

Look at the set $C = \{ p : \pi(p|\underline{X}) \geq k \}$

C is an interval.

Now find k by equating

$$\pi(C|\underline{X}) = 1-\alpha.$$

When the posterior is not in a closed form

run MCMC, draw M MCMC samples of θ.

Let these MCMC samples of θ be

$\theta^{(1)}, \dots, \theta^{(M)}$.

Approximate posterior dist. $\pi(\theta|\underline{X})$ by a discrete dist. that has masses at $\theta^{(1)}, \dots, \theta^{(M)}$ each with prob. $\frac{1}{M}$.

2:00 PM : Monday.

Bayesian ~~confidence~~ credible intervals

HPD credible interval.

$\{\theta : \pi(\theta|\underline{X}) > k\}$ = HPD credible set.

When $\pi(\theta|\underline{X})$ is unimodal, this set will ~~give~~ result in an interval.

$\theta^{(1)}, \ldots, \theta^{(M)}$ be the M MCMC post burn-in samples.

Construct an empirical dist. using $\theta^{(1)}, \ldots, \theta^{(M)}$, i.e. a dist. which puts mass $\frac{1}{M}$ to any $\theta^{(i)}$.

Now one can create HPD credible interval in the following way.

Order $\theta^{(1)}, \ldots, \theta^{(M)}$ in an increasing way

$\theta_*^{(1)} < \cdots\cdots < \theta_*^{(M)}$

look for $\theta_*^{(i)}$ s.t. $\frac{i}{M} = \alpha/2$

Also look for j s.t. $\frac{j}{M} = 1 - \alpha/2$.

then $[\theta_*^{(i)}, \theta_*^{(j)}]$ is the HPD credible interval.

Method of evaluating confidence intervals

lets fix the coverage and look at the length of the confidence intervals ..

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known

$$\left\{ \mu : \bar{X} + \frac{\sigma}{\sqrt{n}} a \le \mu \le \bar{X} + \frac{\sigma}{\sqrt{n}} b \right\}$$

$a = -b = z_{\alpha/2}$.

Thm: If $f(x)$ is unimodal pdf. If the interval $[a, b]$ satisfies

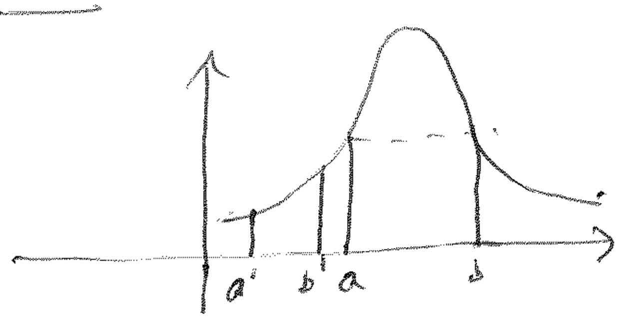
(1) $\int_a^b f(x)\,dx = 1-\alpha$ (2) $f(a) = f(b) > 0$ (3) $a \le x^* \le b$,

where x^* is the mode of $f(x)$. Then $[a, b]$ is the shortest interval that satisfies (1).

Pf: Let $[a', b']$ be any other interval satisfying $b' - a' < b - a$. We will prove $\int_{a'}^{b'} f(x)\,dx < 1-\alpha$.

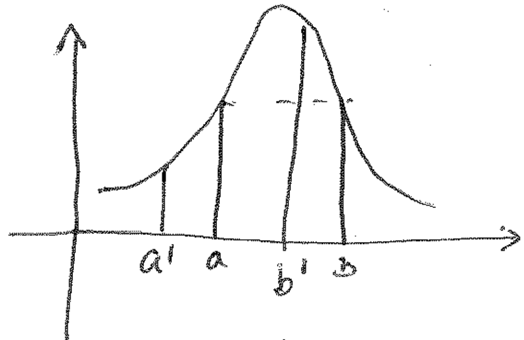
We will prove the result for $a' \le a$. The proof will be similar for $a' > a$. Also for $a' \le a$, consider two cases:

Case I: $b' \le a$, i.e. $a' \le b' \le a < b$



$$\int_{a'}^{b'} f(x)\,dx \le f(a)(b'-a') < f(a)(b-a) < \int_a^b f(x)\,dx = 1-\alpha$$

Case 2 :



$b' > a$ and $a' \leq a \leq b' \leq b$

$$\int_{a'}^{b'} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \left[\int_{a'}^{a} f(x)\,dx - \int_{b'}^{b} f(x)\,dx\right]$$

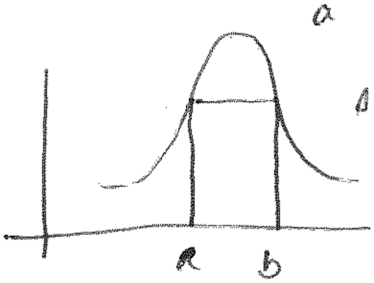
$$\int_{a'}^{a} f(x)\,dx \leq f(a)\,(a-a')\,, \quad \int_{b'}^{b} f(x)\,dx \geq f(b)\,(b-b')$$

$$\Rightarrow \int_{a'}^{b'} f(x)\,dx \leq \int_{a}^{b} f(x)\,dx + f(a)(a-a') - f(b)(b-b')$$

$$= \int_{a}^{b} f(x)\,dx + f(a)\left[(a-a') - (b-b')\right] \quad \left[\text{as } f(a) = f(b)\right]$$

$$= \int_{a}^{b} f(x)\,dx + f(a)\left[\underbrace{(b'-a') - (b-a)}_{<0}\right]$$

$$< \int_{a}^{b} f(x)\,dx = 1-\alpha.$$



s.t. $f(a) = f(b)$.

This is an important result for the location families where the length of the interval $= c(b-a)$.

look at the interval μ for $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\left\{\mu : \bar{X} + \frac{\sigma}{\sqrt{n}} a \leq \mu \leq \bar{X} + \frac{\sigma}{\sqrt{n}} b \right\}$$

the length of the interval is $(b-a)\frac{\sigma}{\sqrt{n}}$.

So, this length can be minimized if one choses $f(b) = f(a)$.

This problem is equivalent to a constrained optimization problem.

minimize $(b-a)$ subject to $\int_a^b f(x) = 1-\alpha$.

Thus b has to be a function of a. Take derivative w.r.t a.

$$\frac{db}{da} - 1 = 0 , \quad \frac{db}{da} f(b) - f(a) = 0$$

$$\Rightarrow f(b) = f(a).$$

Example: Suppose $X \sim$ Gamma (K, β). The quantity $Y = \frac{X}{\beta} \sim$ Gamma$(K, 1)$. Y is a pivot.

A $100(1-\alpha)\%$ CI for β can be obtained in the following way:

$$P\left(a \leq \frac{Y}{\beta} \leq b \right) = 1-\alpha \Leftrightarrow P\left(\frac{Y}{b} \leq \beta \leq \frac{Y}{a} \right) = 1-\alpha$$

$\Rightarrow \left[\frac{Y}{b}, \frac{Y}{a}\right]$ is a $100(1-\alpha)\%$ CI for β.

This interval obtained from the pivotal

quantity is known as a pivotal interval.

Here any pivotal interval using the pivot $\frac{Y}{\beta}$ should be of the form $\left[\frac{Y}{b}, \frac{Y}{a}\right]$

Qn: Find a, b that gives minimum length of such an interval with coverage $1-\alpha$.

length $= Y\left[\frac{1}{a} - \frac{1}{b}\right]$

Thus the optimization problem is

minimize $\frac{1}{a} - \frac{1}{b}$ s.t. $\int_a^b f(x)\,dx = 1-\alpha$.

$$-\frac{1}{a^2} + \frac{1}{b^2}\frac{db}{da} = 0 \quad \text{and} \quad f(b)\frac{db}{da} - f(a) = 0$$

$$\Rightarrow \frac{db}{da} = \frac{b^2}{a^2} \Rightarrow f(b)\frac{b^2}{a^2} = f(a) \Rightarrow b^2 f(b) = a^2 f(a).$$

Thus we have to find a, b in the density Gamma$(K, 1)$ s.t. $a^2 f(a) = b^2 f(b)$.

This is the criterion to find the shortest interval among all pivotal intervals using the pivot $\frac{Y}{\beta}$.

Is there any connection between optimal tests and optimal intervals.

Note that probability of true coverage for an interval $C(\underline{X})$ is given by $P_\theta(\theta \in C(\underline{X}))$

The probability of false coverage is given by

$P_\theta(\theta' \in C(\underline{X})),\ \theta' \neq \theta,$ if $C(\underline{X}) = [L(\underline{X}), U(\underline{X})]$

$P_\theta(\theta' \in C(\underline{X})),\ \theta < \theta',$ if $C(\underline{X}) = (-\infty, U(\underline{X})]$

$P_\theta(\theta' \in C(\underline{X})),\ \theta > \theta',$ if $C(\underline{X}) = [L(\underline{X}), \infty)$.

Note that ~~coverage~~ the false coverage will be big if we unnecessarily cover unimportant θ'. Thus we want an interval (or a set) which minimizes the prob. of false coverage.

UMA confidence set: A $(1-\alpha)$ confidence set that minimizes prob. of false coverage over a class of $(1-\alpha)$ confidence sets is called a uniformly most accurate (UMA) confidence set.

Thm: $X_1, \dots, X_n \overset{iid}{\sim} f_\theta(x)$, where θ is real valued. For each $\theta_0 \in \Theta$, let $A^*(\theta_0)$ be the UMP level α acceptance region of a test $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$. Let $C^*(\underline{X})$ be the $(1-\alpha)$ confidence set formed by inverting the UMP acceptance regions. Then for any $(1-\alpha)$ confidence set $C(\underline{X})$,

$$P_\theta(\theta' \in C^*(\underline{X})) \leq P_\theta(\theta' \in C(\underline{X})) \quad \forall\, \theta' < \theta.$$

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$,

$C(\underline{X}) = \left\{ \mu : \mu > \bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$ is the $100(1-\alpha)\%$ CI for μ obtained by inverting the UMP test for $H_0: \mu = \mu_0$ vs. $H_1: \mu > \mu_0$.

$C(\underline{X})$ has the lowest prob. of false coverage.

Is there any optimality guarantee with UMPU?

Definition: A $(1-\alpha)$ confidence set $C(\underline{X})$ is called unbiased if $P_\theta(\theta' \in C(\underline{X})) \le 1-\alpha \;\; \forall\, \theta' \ne \theta$.

$P_\theta(\theta \in C(\underline{X})) = 1-\alpha$.

By inverting acceptance region of a UMP unbiased (UMPU) test one obtains an unbiased interval.

$H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$

Thm: $X \sim f_\theta$, θ is a real valued parameter.
Let $C(X) = [L(X), U(X)]$ be the confidence interval for θ. If $L(x), U(x)$ are both increasing functions of x, then for any θ':

$$E_{\theta'}[\text{length}(C(X))] = \int_{\theta \ne \theta'} P_{\theta'}(\theta \in C(X))\, d\theta.$$

$$E_{\theta'}[\text{length}(C(\underline{X}))] = \int_{\mathcal{X}} \text{length}[C(\underline{x})]\, f_{\theta'}(x)\, dx$$

$$= \int_{\mathcal{X}} [U(x) - L(x)]\, f_{\theta'}(x)\, dx$$

$$= \int_{\mathcal{X}} \left[\int_{L(x)}^{U(x)} d\theta \right] f_{\theta'}(x)\, dx$$

$$= \int_{\theta \in \Theta} \left[\int_{U^{-1}(\theta)}^{L^{-1}(\theta)} f_{\theta'}(x)\, dx \right] d\theta$$

$$= \int_{\theta \in \Theta} P_{\theta'}\left(U^{-1}(\theta) \le X \le L^{-1}(\theta) \right) d\theta$$

$$= \int_{\theta \in \Theta} P_{\theta'}\left(L(X) \le \theta \le U(X) \right) d\theta$$

$$= \int_{\theta \in \Theta} P_{\theta'}\left(\theta \in C(X) \right) d\theta$$

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