Assignment #3
Problem #2

Given Information:

- $X_1, \dots, X_n \sim N(\theta, a\theta^2)$
- a is a known constant
- $\theta > 0$

Goals:

- Show $(\bar{X}, S^2)$ is a sufficient statistic for θ
- Show the family of distributions is not complete

Work:

$$f_\theta(x_1, \dots, x_n) = \left(\frac{1}{\sqrt{2\pi(a\theta^2)^2}}\right)^n exp\left\{-\frac{(\sum_{i=1}^n x_i^2 - 2\theta\sum_{i=1}^n x_i - n\theta^2)}{2(a\theta^2)^2}\right\}$$
$$= \left(\frac{1}{a\theta^2\sqrt{2\pi}}\right)^n exp\left\{-\frac{1}{2a^2\theta^4}\sum_{i=1}^n x_i^2 + \frac{1}{a^2\theta^3}\sum_{i=1}^n x_i - \frac{n}{2a^2\theta^2}\right\}$$

We can see $T(X) = \left(\sum_{i=1}^n x_i^2\ ,\ \sum_{i=1}^n x_i\right)$ is a two-dimensional sufficient statistic for θ. We also know any one-to-one function composed of sufficient statistics is itself a sufficient statistic. Two such one-to-one functions are $\bar{X}$ and S^2. Therefore, $(\bar{X}, S^2)$ is a sufficient statistics for θ.

To show $(\bar{X}, S^2)$ is not a complete sufficient statistics, recall that the Normal distribution is of the Exponential Families. Hence, we can identify

$$w_1 = \frac{1}{a^2\theta^3} \text{ and } w_2 = -\frac{1}{2a^2\theta^4}$$

However, the parameter space (θ^3, θ^4) does not contain a two-dimensional open set.
Therefore, $(\bar{X}, S^2)$ is not complete.