AMS 205B - ASSIGNMENT 3 DRAFT

Question 1 (CB 6.10)

By definition, a sufficient statistic is not complete if $\exists$ a function $g()$ such that $E(g(T(X))) = 0 \ \forall \theta$ and $g(T(X)) \neq 0$.

The minimal sufficient statistic as found in class is $(X_{(1)}, X_{(n)})$.

Looking at the distribution of $X_{(1)}, X_{(n)}$ we have:

$$P(X_{(1)} \leq x) = 1 - (1 - P(X \leq x))^n$$
$$= 1 - (1 - (x - \theta))^n$$

$$P(X_{(n)} \leq x) = P(X \leq x)^n$$
$$= (x - \theta)^n$$

$z = 1 - x + \theta \qquad x = 1 + \theta - z$

Taking the expected values, we have:

- $E(X_{(1)}) = \int_{\theta}^{\theta+1} n(1 - x + \theta)^{n-1} x \, dx = \int_0^1 n z^{n-1} (1 + \theta - z) \, dz = z^n (1+\theta) - \frac{n}{n+1} z^{n+1} \Big|_0^1$
- $E(X_{(n)}) = \int_{\theta}^{\theta+1} n(x - \theta)^{n-1} x \, dx = \int_0^1 n z^{n-1} (z + \theta) \, dz = \frac{n z^{n+1}}{n+1} + z^n \theta \Big|_0^1 = \frac{n}{n+1} + \theta$

$$\therefore g(T(x)) = X_{(n)} - X_{(1)} + \frac{2n}{n+1} - 1 = X_{(n)} - X_{(1)} + \frac{n-1}{n+1} \neq 0$$

but $E(g(T(X))) = 0 \ \forall \theta$.

Hence, $T(x) = (X_{(1)}, X_{(n)})$ is not complete statistic for θ.

Question 5

Part a.

$E(T(X_1, \ldots, X_{n+1})) = 1 \cdot P_p(\sum_{i=1}^{n} X_i > X_{n+1}) + 0 = P_p(\sum_{i=1}^{n} X_i > X_{n+1}) = h(p)$

$\therefore T(X_1, \ldots, X_{n+1})$ is an unbiased estimator of $h(p)$.

Part b.

First we find a sufficient statistic for p. We have shown in class that $\sum_{i=1}^{n+1} X_i$ is a complete sufficient statistic.

Now using Rao-Blackwell, we can find that UMVUE $= E(T \mid \sum_{i=1}^{n} X_i = s)$

$$E(T \mid \sum_{i=1}^{n+1} X_i = s) = P(T=1 \mid \sum_{i=1}^{n} X_i = s) = \frac{P(T=1, \sum_{i=1}^{n+1} X_i = s)}{P(\sum_{i=1}^{n+1} X_i = s)}$$

- If $s=0$, then $P(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i = 0) = 0$
- If $s \geq 3$, then $P(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i \geq 3) = P(\sum_{i=1}^{n+1} X_i \geq 3)$

  since $\sum_{i=1}^{n} X_i \geq 2$ and $X_{n+1} = \{0, 1\}$

- If $s=1$, then

$$P(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i = 1) = \underbrace{n}_{\substack{\text{one of } X_i = 1 \\ i=1,\dots,n}} p(1-p)^{n-1} \overbrace{(1-p)}^{X_{n+1}=0}$$

$$P(\sum_{i=1}^{n+1} X_i = 1) = (n+1)\, p(1-p)^n$$

- If $s=2$, then

$$P(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i = 2) = \underbrace{\binom{n}{2}}_{\substack{\text{two } X_i = 1 \\ i=1,\dots,n}} p^2(1-p)^{n-2} \overbrace{(1-p)}^{X_{n+1}=0}$$

$$P(\sum_{i=1}^{n+1} X_i = 2) = \frac{n(n+1)}{2} p^2(1-p)^{n-1}$$

In sum, the UMVUE is: $t = \sum_{i=1}^{n+1} X_i$

$$\phi(t) = \begin{cases} 0 & \text{if } t=0 \\ n(n+1)^{-1} & \text{if } t=1 \\ (n-1)(n+1)^{-1} & \text{if } t=2 \\ 1 & \text{if } t \geq 3 \end{cases}$$