Question 1 (CB 6.10)

By definition, a sufficient statistic is not complete if \( \exists \) a function \( g() \) such that \( E(g(T(X))) = 0 \ \forall \ \Theta \) and \( g(T(X)) \neq 0 \).

The minimal sufficient statistic as found in class is \((X_0, X_\infty)\).

Looking at the distribution of \( X_0, X_\infty \) we have:

\[
P(X_0 \leq x) = 1 - (1 - P(X \leq x))^n
= 1 - (1 - (x - \Theta))^n
\]

\[
P(X_\infty \leq x) = P(X \leq x)^n
= (x - \Theta)^n\]

Taking the expected values, we have:

\[
E(X_0) = \int_0^{\infty} n (1 - x + \Theta)^{n-1} x \, dx = \int_0^{\infty} n z^{n-1} (1 + \Theta - z) \, dz = \int_0^{\infty} z^n (1 + \Theta) - n z^n \, dz
\]

\[
E(X_\infty) = \int_0^{\infty} n (x - \Theta)^{n-1} x \, dx = \int_0^{\infty} n z^{n-1} (z + \Theta) \, dz = \int_0^{\infty} n z^n + z^n \Theta \, dz
\]

\[
g(T(X)) = X_\infty - X_0 + 2n - 1 = X_\infty - X_0 + \frac{n-1}{n+1} \neq 0
\]

but \( E(g(T(X))) = 0 \ \forall \ \Theta \).

Hence, \( T(X) = (X_0, X_\infty) \) is not complete statistic for \( \Theta \).
**Question 5**

**Part a.**

\[ E(T(X_1, \ldots, X_{n+1})) = 1 \cdot P_p(\sum_{i=1}^{n+1} X_i > X_{n+1}) + 0 = P(T \geq X_{n+1}) = h(p) \]

\[ \therefore T(X_1, \ldots, X_{n+1}) \text{ is an unbiased estimator of } h(p). \]

**Part b.**

First we find a sufficient statistic for \( p \). We have shown in class that \( \sum_{i=1}^{n+1} X_i \) is a complete sufficient statistic.

Now using Rao–Blackwell, we can find that UMVUE:\n
\[ E(T | \sum_{i=1}^{n} X_i = s) \]

\[ E(T | \sum_{i=1}^{n+1} X_i = s) = P(T=1 | \sum_{i=1}^{n} X_i = s) = \frac{P(T=1, \sum_{i=1}^{n+1} X_i = s)}{P(\sum_{i=1}^{n+1} X_i = s)} \]
• If \( s = 0 \), then \( P(\sum_{i=1}^{n} \bar{X}_i > X_{n+1}, \sum_{i=1}^{n+1} \bar{X}_i = 0) = 0 \)

• If \( s \geq 3 \), then \( P(\sum_{i=1}^{n} \bar{X}_i > X_{n+1}, \sum_{i=1}^{n+1} \bar{X}_i \geq 3) = P(\sum_{i=1}^{n+1} \bar{X}_i \geq 3) \)
  since \( \sum_{i=1}^{n} X_i \geq 2 \) and \( X_{n+1} \in \{0, 1\} \)

• If \( s = 1 \), then one of \( \bar{X}_i = 1 \)
  \[ P(\sum_{i=1}^{n} \bar{X}_i > X_{n+1}, \sum_{i=1}^{n+1} \bar{X}_i = 1) = \frac{n}{(n+1)} p (1-p)^{n-1} (1-p) \]

  \[ P(\sum_{i=1}^{n+1} \bar{X}_i = 1) = (n+1) p (1-p)^n \]

• If \( s = 2 \), then two \( \bar{X}_i = 1 \)
  \[ P(\sum_{i=1}^{n} \bar{X}_i > X_{n+1}, \sum_{i=1}^{n+1} \bar{X}_i = 2) = \binom{n}{2} p^2 (1-p)^{n-2} (1-p) \]

  \[ P(\sum_{i=1}^{n+1} \bar{X}_i = 2) = \frac{n(n+1)}{2} p^2 (1-p)^{n-1} \]

In sum, the UMVUE is: \( t = \sum_{i=1}^{n+1} \bar{X}_i \)

\[ \Phi(t) = \begin{cases} 
0 & \text{if } t = 0 \\
\frac{n(n+1)}{2} & \text{if } t = 1 \\
\frac{(n-1)(n+1)}{2} & \text{if } t = 2 \\
1 & \text{if } t \geq 3
\end{cases} \]