Sam Leonard
205B Solutions

HW 1, #4

Binomial case:

$E[e^{tX}] = \sum_{i=0}^{n} \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i} e^{tx_i}$

$= \sum_{i=0}^{n} \binom{n}{x_i} (pe^t)^{x_i}(1-p)^{n-x_i}$

And by the binomial theorem, we have $E[e^{tX}] = \phi_X(t) = (pe^t + (1-p))^n$

Poisson case:

$E[e^{tX}] = \sum_{x=0}^{\infty} (\frac{e^{-\lambda}\lambda^x}{x!})(e^{tx})$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

The sum is in the form of the series expansion for $e^x$. So:

$E[e^{tX}] = \phi_X(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t - 1)}$

HW 3, #8

First find a complete, sufficient statistic:

$f_\theta(x_1, ..., x_n) = (\frac{1}{2\theta})^n$ when $\theta < x_i < \theta$ for all $i$ and 0 otherwise.

If $\max|X_i| < \theta$, we know that $-\theta < x_i < \theta$ for all $i$.

So $f_\theta(x_1, ..., x_n) = (\frac{1}{2\theta})^n I(\max|X_i| < \theta)$

$\max|X_i|$ is a sufficient statistic by the Factorization Theorem. Is it complete?

$X_i \sim U(-\theta, \theta)$, so $|X_i| \sim U(0, \theta)$.

And then $Pr(\max|X_n| \leq y) = (\int_0^t \frac{1}{\theta} dx)^n = (\frac{y}{\theta})^n$.

So the distribution of $Y = \max|X_i|$ is $f(y) = \frac{ny^{n-1}}{\theta^n}$

Assume $E[g_\theta(Y)] = 0$ for all $\theta$ and an arbitrary function $g$.

This means $\int_0^\theta g(y) \frac{ny^{n-1}}{\theta^n} dy = 0$ for all $\theta$.

$\implies \int_0^\theta g(y) y^{n-1} dy = 0 \ \forall \ \theta$

$\implies \frac{d}{d\theta} \int_0^\theta g(y) y^{n-1} dy = 0 \ \forall \ \theta$

$\implies g(\theta)\theta^{n-1} = 0 \ \forall \ \theta$.

This is true only if $g(\theta) = 0$ for all $\theta$. So the statistic $T(\underline{x}) = \max|X_i|$ is complete and sufficient.

If we can find a function $\phi$ such that $E[\phi(T)] = \theta$ then by Lehmann-Scheffe $\phi(T)$ will be UMVUE.

First look at the expected value of $T$:

$E[T] = \int_0^\theta y \frac{ny^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \int_0^\theta y^n dy = \frac{n}{\theta^n} [\frac{y^{n+1}}{n+1}]_0^\theta = \frac{n\theta}{n+1}$

So this means if we set $\phi(T) = \frac{n+1}{n} T$, $E[\phi(T)] = \theta$.

And by Lehmann-Scheffe $\frac{n+1}{n} \max|X_i|$ is the UMVUE of $\theta$.