4. • $\psi_X(t) = E[\exp(tX)] = \sum_{x=0}^{n} \exp(tx) \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$

$= (e^t p + 1 - p)^n$ for $X \sim \text{Binomial}(n, p)$

• $\psi_X(t) = E[\exp(tX)] = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\lambda^x \exp(-\lambda)}{x!}$

$= \sum_{x=0}^{\infty} \frac{\exp(-\lambda)}{x!} \cdot (e^t\lambda)^x = \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} (e^{-e^t\lambda}) \cdot e^{e^t\lambda} \cdot e^{-\lambda}$

$= e^{(e^t-1)\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} e^{-e^t\lambda} = e^{(e^t-1)\lambda}$

for $X \sim \text{Pois}(\lambda)$

5 $X, Y \overset{iid}{\sim} N(0,1)$ $Z = \min(X, Y)$

• $F_Z(z) = Pr(Z \le z) = 1 - Pr(Z > z) = 1 - Pr(X > z, Y > z)$

$= 1 - Pr(X > z) Pr(Y > z)$ (∵ X & Y are independant)

$= 1 - Pr(X > z)^2$ (∵ X & Y are identical)

$= (1 - Pr(X > z))(1 + Pr(X > z)) = Pr(X \le z)(1 + Pr(X > z))$

• $F_{Z^2}(a) = Pr(Z^2 \le a) = Pr(-\sqrt{a} \le Z \le \sqrt{a})$

$= F_Z(\sqrt{a}) - F_Z(-\sqrt{a})$

$= Pr(X \le \sqrt{a})(1 + Pr(X > \sqrt{a})) - Pr(X \le -\sqrt{a})(1 + Pr(X > -\sqrt{a}))$

※ letting $Pr(X \le \sqrt{a}) = \alpha$, then $Pr(X \ge \sqrt{a}) = 1 - \alpha$, $Pr(X \le -\sqrt{a}) = 1 - \alpha$

$Pr(X > -\sqrt{a}) = \alpha$

∴ $F_{Z^2}(a) = \alpha(2 - \alpha) - (1 - \alpha)(1 + \alpha) = 2\alpha - \alpha^2 - 2 + \alpha^2$

$= 2\alpha - 2 = 2Pr(X \le \sqrt{a}) - 2 = 2F_X(\sqrt{a}) - 2$

$f_{Z^2}(a) = 2f_X(\sqrt{a}) \cdot \frac{1}{2\sqrt{a}} = \frac{1}{\sqrt{a}} f_X(\sqrt{a})$

• $f_{Z^2}(z) = \frac{1}{\sqrt{z}} f_X(\sqrt{z}) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{z}} \exp(-\frac{z}{2})$

∴ $Z^2 \sim \chi^2_1$