8 As a generalization to the previous excersize, let $X_1 \dots X_n$ be iid w/ p.d.f. $f_X(x) = \begin{cases} a/\theta^a \cdot x^{a-1}, & \text{if } 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$

Let $X_{(1)} < \dots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(2)}$, $X_{(2)}/X_{(3)}$, $\dots X_{(n-1)}/X_{(n)}$ and $X_{(n)}$ are mutually independent random variable(s). Find the distribution of them.

Let's start by finding the distribution of them. I am not sure if Raj means to find each individually or the joint p.d.f. So let's do both! :)

First, we know $f_X(x) = \begin{cases} a/\theta^a \, x^{a-1}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$

$$\text{So, } F_X(x) = \int_0^x a/\theta^a \, x^{a-1} \, dx$$
$$= a/\theta^a \cdot 1/a \, x^a \Big|_0^x$$
$$= 1/\theta^a \, x^a, \quad 0 < x < \theta$$

Since we are trying to prove $X_{(1)}/X_{(2)}$, $X_{(2)}/X_{(3)} \dots X_{(n-1)}/X_{(n)}$, $X_{(n)}$ are independent, it will be easy to find their pdf's after we find the joint p.d.f. of them together.

$$f_{X_{(1)} \dots X_{(n)}}(x_1 \dots x_n) = \begin{cases} n! \, f_X(x_1) \dots f_X(x_n), & -\infty < x_1 < \quad < x_n < \infty \\ 0, & \text{o.w} \end{cases}$$

which comes from (5.4.7) in Casella-Berger

$$= \begin{cases} n! \, a^n \, \theta^{-an} (x_1 \cdot \dots \cdot x_n)^{a-1}, & 0 < x_1 < \dots < x_n < \theta \\ 0, & \text{o.w.} \end{cases}$$

This comes from the idea that there are n! permutations to order $X_1 \dots X_n$.

#8 cont.

Define $y_1 = X_{(1)}/X_{(2)},\ y_2 = X_{(2)}/X_{(3)},\ \dots,\ y_{n-1} = X_{(n-1)}/X_{(n)},\ y_n = X_{(n)}$

$\Rightarrow X_{(n)} = y_n,\ X_{(n-1)} = y_{n-1}y_n,\ X_{(n-2)} = y_{n-2}y_{n-1}y_n,\ \dots\ X_{(1)} = y_1 \cdots y_n$

$$\text{So } |J| = \begin{vmatrix} dX_{(1)}/dy_1 & \cdots & dX_{(1)}/dy_n \\ \vdots & & \vdots \\ dX_{(n)}/dy_1 & \cdots & dX_{(n)}/dy_n \end{vmatrix} = \begin{vmatrix} y_2 \cdots y_n & y_1 y_3 \cdots y_n & y_1 y_2 y_4 \cdots y_n & \cdots & y_1 \cdots y_{n-1} \\ 0 & y_3 \cdots y_n & y_2 y_4 \cdots y_n & \cdots & y_2 \cdots y_{n-1} \\ \vdots & 0 & y_4 \cdots y_n & \cdots & y_3 \cdots y_{n-1} \\ & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= |y_2 y_3^2 y_4^3 \cdots y_n^{n-1} + 0|$$

$$= y_2 y_3^2 y_4^3 \cdots y_n^{n-1}$$

Now $f_{y_1 \dots y_n}(y_1 \dots y_n) = n!\, a^n \theta^{-an} (y_1 \cdots y_n)^{a-1} (y_2 \cdots y_n)^{a-1} \cdots (y_n)^{a-1} |J|$

$$= n!\, a^n \theta^{-an} y_1^{a-1} y_2^{2a-2} y_3^{3a-3} \cdots y_n^{na-n} (y_2 y_3^2 \cdots y_n^{n-1})$$

$$= n!\, a^n \theta^{-an} y_1^{a-1} y_2^{2a-1} y_3^{3a-1} \cdots y_n^{na-1},\ 0 < y_i < 1$$

So, finally we have

$f_{y_1 \dots y_n}(y_1 \dots y_n) = n!\, a^n \theta^{-an} y_1^{a-1} y_2^{2a-1} \cdots y_n^{na-1},\ 0 < y_i < 1$

where $y_1 = X_{(1)}/X_{(2)},\ y_2 = X_{(2)}/X_{(3)},\ \dots,\ y_{n-1} = X_{(n-1)}/X_{(n)},\ y_n = X_{(n)}$

Since $f_{y_1 \dots y_n}(y_1 \dots y_n)$ factors, then $y_1 \dots y_n$ are independent.

Now to find the p.d.f's of $y_1 \dots y_n$ separately

$f_{y_1}(y_1) = \int_0^1 \cdots \int_0^1 n!\, a^n \theta^{-an} y_1^{a-1} y_2^{2a-1} \cdots y_n^{na-1}\, dy_2 \dots dy_n$

which must be in the form $f_{y_1}(y_1) = c y_1^{a-1}$, b/c all other variables, $y_2 \dots y_n$ must factor out of the function

To find c, integrate to equal 1.

$\Rightarrow \int_0^1 c y_1^{a-1} dy_1 = c/a\, y_1^a \Big|_0^1 = c/a = 1 \Rightarrow c = a$

So, $f_{y_1}(y_1) = a y_1^{a-1},\ 0 < y_1 < 1$, where $y_1 = X_{(1)}/X_{(2)}$

similarly $f_{y_i}(y_i) = ia\, y_i^{ia-1},\ 0 < y_i < 1$, where $y_i = X_{(i)}/X_{(i+1)}$

Also, as proven in #7 $f_{X_{(n)}}(x_n) = \frac{na}{\theta^{na}} x_n^{na-1},\ 0 < x_n < \theta$

Let X take values 0, 1, 2 w/ probability p, 3p, 4p. Determine if the family of distributions of X is complete.

X	0	1	2
prob	p	3p	4p

$\Rightarrow 1 = 8p \Rightarrow p = 1/8$

X	0	1	2
prob	1/8	3/8	1/2

$$E(g(x)) = p \cdot g(0) + 3p \cdot g(1) + 4p \cdot g(2)$$
$$= 1/8\, g(0) + 3/8\, g(1) + 1/2\, g(2)$$

let $g(0) = 0$, $g(1) = -4/3$, & $g(2) = 1$

Then $E(g(x)) = 0, \forall$ but $g(x) \neq 0$