AMS 205B <u>Homework 3 Question 4</u>

Let $X_1, \cdots, X_n$ be a random sample from the pdf $f_\mu(x) = e^{-(x-\mu)}$, $-\infty < \mu < x < \infty$.

Show that $X_{(1)}$ and $S^2$ are independent.

- Show that $S^2$ is an ancillary statistic :

$$S^2(x) = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

$$= \frac{1}{n-1} \sum_{i=1}^n (x_i + \theta - \theta - \bar{x})^2$$

$$= \frac{1}{n-1} \sum_{i=1}^n [(x_i + \theta) - (\bar{x} + \theta)]^2$$

$$= S^2(x + \theta)$$

$S^2(x)$ is from location family and hence $S^2(x)$ is an ancillary statistic .

- Show that $X_{(1)}$ is sufficient and complete :

$\boxed{\text{Sufficiency}}$

Joint pdf of the random sample is

$$f_\mu(x_1, \cdots, x_n) = e^{\substack{-\sum_{i=1}^n x_i + n\mu \\ \cancel{-\sum_{i=1}^n x_i\, n\mu}}} \mathbb{1}\{\mu < x_1, \cdots, x_n < \infty\}$$

$$= e^{-\sum_{i=1}^n x_i + n\mu} \mathbb{1}\{x_{(1)} > \mu\}$$

Set

$$h(\underline{x}) = e^{-\sum_{i=1}^{n} x_i}$$

$$g(T(\underline{x}), \mu) = e^{n\mu} \mathbb{1}\{x_{(1)} > \mu\},$$

then by the statement of Factorization Theorem therefore $T(X) = X_{(1)}$ is sufficient statistic for $\mu$.

$\boxed{\text{Completeness}}$

$$\begin{aligned}
Pr(T(\underline{X}) \le t) &= Pr(X_{(1)} \le t) \\
&= 1 - Pr(X_{(1)} > t) \\
&= 1 - Pr(X_1, \cdots, X_n > t) \\
&= 1 - Pr(X_1 > t) \cdots Pr(X_n > t) \\
&= 1 - [Pr(X_i > t)]^n \\
&= 1 - [1 - Pr(X_i \le t)]^n \\
&= 1 - [1 - (1 - e^{-(t-\mu)})]^n \\
&= 1 - e^{-n(t-\mu)}
\end{aligned}$$

$$\Rightarrow f_{X_{(1)}}(t) = n e^{-n(t-\mu)}. \quad \infty > t > \mu$$

$$E_\mu[g(T(\underset{\sim}{X}))] = 0 \qquad \forall \mu$$

$$= E_\mu[g(X_{(1)})]$$

$$= \int_\mu^\infty g(t)\, n e^{-n(t-\mu)}\, dt = 0$$

$$\Rightarrow \int_\mu^\infty g(t)\, n e^{-n(t-\mu)}\, dt = 0$$

$$\frac{d}{d\mu} \int_\mu^\infty g(t)\, n e^{-n(t-\mu)}\, dt = 0$$

By Newton-Leibniz Rule & Product Rule & Interchangebility.

$$0 - g(\mu)\, n e^{-n(\mu-\mu)} = 0$$

$$g(\mu) = 0 \qquad \forall \mu$$

Hence by the definition of complete statistic

$X_{(1)}$ is complete statistic.

In this case, $S^2$ is ancillary statistic

and $X_{(1)}$ is complete sufficient statistic

for $\mu$,

by Basu's Theorem, $X_{(1)}$ and $S^2$ are

independent.