- Show that the morginal distribution of any K of the Xs is the same as:

$$P(X_1 = x_1, \dots, X_K = x_K) = \int_0^1 p^t (1-p)^{K-t}\, dp, \quad t = \sum_{i=1}^{K} x_i$$

Using the law of total probability:

$$P(X_1 = x_1, \dots, X_K = x_K) = \int_0^1 P(X_1 = x_1, \dots, X_K = x_K \mid p) \cdot \pi(p)\, dp$$

Since $X_1, \dots, X_K$ are conditional independent given p

$$= \int_0^1 P(X_1 = x_1 \mid p) \cdot \ldots \cdot P(X_K = x_K \mid p) \cdot \pi(p)\, dp$$

$X_i \mid p \sim Ber(p)$ and $p \sim Unif(0,1)$

$$= \int_0^1 p^{x_1}(1-p)^{1-x_1} \cdot \ldots \cdot p^{x_K}(1-p)^{1-x_K}\, dp$$

$$= \int_0^1 p^{\sum_{i=1}^{K} x_i} (1-p)^{K - \sum x_i}\, dp$$

$$= \int_0^1 p^t (1-p)^{K-t}\, dp$$

$$= \int_0^1 p^{t+1-1} (1-p)^{K-t+1-1}\, dp$$

Using the fact that:

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} = 1$$

(Beta distribution)

$$= B(t+1, K-t+1)$$

$$= \frac{\Gamma(t+1)\,\Gamma(K-t+1)}{\Gamma(K+2)} = \frac{t!\,(K-t)!}{(K+1)!}$$

K, t positive integers

$\rightarrow$ Since $P(X_1 = x_1, \dots, X_K = x_K) = \dfrac{t!\,(K-t)!}{(K+1)!} = P(X_{\pi(1)} = x_{\pi(1)}, \dots, X_{\pi(K)} = x_{\pi(K)})$

for any permutation $\pi(1), \dots, \pi(K)$, $X_1, \dots, X_K$ are exchangeable

- Show that marginally $P(X_1 = x_1, \dots, X_n = x_n) \neq P(X_1 = x_1) \dots P(X_n = x_n)$ hence Xs are not iid

Using the same principle:

$$P(X_i = x_i) = \int_0^1 P(X_i = x_i \mid p)\, \pi(p)\, dp \qquad \forall i = 1, \dots, n$$

$$= \int_0^1 p^{x_i} (1-p)^{1-x_i}\, dp \qquad \forall i = 1, \dots, n$$

$$= \int_0^1 p^{x_i + 1 - 1} (1-p)^{1 - x_i + 1 - 1}\, dp$$

$$= B(x_i + 1, 2 - x_i) = \frac{\Gamma(x_i + 1)\,\Gamma(2 - x_i)}{\Gamma(3)}$$

$$= \frac{x_i!\,(1 - x_i)!}{2}$$

then:

$$\prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} \frac{x_i!\,(1 - x_i)!}{2} \neq \frac{t!\,(n-t)!}{(n+1)!}$$

From previous part

with $t = \sum_{i=1}^{n} x_i$

then $P(X_1 = x_1, \dots, X_n = x_n) \neq P(X_1 = x_1) \dots P(X_n = x_n)$

thus $X_1, \dots, X_n$ are not iid

ASSIGNMENT 2 — NUMBER 4

(4) Let $X_1, \ldots, X_n$ be a random sample from the pdf:

$$f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} & \text{if } \mu < x < \infty, \ 0 < \sigma < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find a two-dimensional sufficient statistic for μ, σ

$$f_{\mu,\sigma}(X_1, \ldots, X_n \mid \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{(x_i-\mu)}{\sigma}} \cdot 1_{\{x_i > \mu\}}$$

$$= \left(\frac{1}{\sigma}\right)^n e^{-\frac{\Sigma(x_i-\mu)}{\sigma}} \cdot 1_{\{\min(x_i) > \mu\}}$$

$$= \left(\frac{1}{\sigma}\right)^n e^{-\frac{\Sigma x_i}{\sigma}} e^{\frac{\mu}{\sigma}} \cdot 1_{\{\min(x_i) > \mu\}}$$

$$= h(\underline{x}) \cdot g(T(\underline{x}), \underline{\theta})$$

with $h(\underline{x}) = 1$

$$g(T(\underline{x}), \theta) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{\Sigma x_i}{\sigma}} e^{\frac{\mu}{\sigma}} 1_{\{\min(x_i) > \mu\}}$$

where $T(\underline{x}) = (\Sigma x_i, X_{(1)})$

By the factorization theorem $T(\underline{x})$ is a two-dimensional sufficient statistic for μ, σ.