# Assignment 4

February 6, 2017

1. Using change of variable theorem calculate the value of $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx$.

2. $\boldsymbol{X} = (X_1, ..., X_k) \sim Dirichlet(\alpha_1, ..., \alpha_k)$, $X_k = 1 - \sum_{i=1}^{k-1} X_i$, if the joint p.d.f has the form

$$f_{\alpha_1,...,\alpha_k}(x_1, ..., x_{k-1}) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left[\prod_{i=1}^{k-1} x_i^{\alpha_i - 1}\right] (1 - \sum_{i=1}^{k-1} x_i)^{\alpha_k - 1}, \ x_1, ..., x_{k-1} > 0, \ \sum_{i=1}^{k-1} x_i < 1.$$

   Prove that if $Y_i \sim Gamma(\alpha_i, \theta)$, $V = \sum_{i=1}^{k} Y_i$, then $\left(\frac{Y_1}{V}, ..., \frac{Y_k}{V}\right) \sim Dirichlet(\alpha_1, ..., \alpha_k)$.

3. Using the Central Limit theorem show that $\sum_{i=0}^{n} \exp(-n)\frac{n^i}{i!} \to \frac{1}{2}$.

4. Use the result that if $X_n \to c$ in probability then $g(X_n) \to g(c)$ in probability where $g$ is continuous in $c$. Let $\{Z_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$ be two sequences such that $E[Z_n] = E[Y_n] = 0$ and $E[Z_n^2] = E[Y_n^2] = \sigma^2$. Prove using the above result, Central Limit theorem, Weak Law of Large number and Slutsky's theorem that

$$\frac{Z_1 Y_1 + \cdots + Z_n Y_n}{Z_1^2 + \cdots + Z_n^2} \to N(0, 1), \text{ in distribution.}$$

5. Let $X$,..., $X_n$ be a random sample with cumulative density function (c.d.f) of $X$ is $F(x)$, i.e. $P(X \leq x) = F(x)$. Prove that $P(X_{(j)} \leq x) = \sum_{r=j}^{n} \binom{n}{r} (F(x))^r (1 - F(x))^{n-r}$.

6. Using Delta theorem prove that

- If $X_1, ..., X_n \overset{iid}{\sim} Ber(p), 0 < p < 1$ then $\sqrt{n}[\sin^{-1}(\sqrt{\bar{X}_n}) - \sin^{-1}(\sqrt{p})] \to N(0, 1/4)$.

- If $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\sqrt{n}[\log(S_n) - \log(\sigma^2)] \to N(0, 2)$, where $S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$.

Notice that in all these cases the asymptotic variance is free of any parameter. Such a transformation is known as *variance stabilizing* transformation of the statistic.

7. If $X$ follows a multi-parameter exponential family density and $g$ is any differentiable function s.t. $E_{\boldsymbol{\theta}}|g'(X)| < \infty$ then

$$E\left[\left\{\frac{h'(X)}{h(X)} + \sum_{i=1}^{k} w_i(\boldsymbol{\theta})T_i'(X)\right\} g(X)\right] = -E[g'(X)],$$

provided the support of $X$ is $(-\infty, \infty)$.

8. Let $X_1, ..., X_n$ be iid geometric distribution with $P_\theta(X = x) = \theta(1-\theta)^{x-1}$, $x = 1, 2, ...$, $0 < \theta < 1$. Show that $\sum_{i=1}^{n} X_i$ is sufficient and complete for $\theta$.

9. Let $(X_1, X_2, X_3, X_4) \sim Multinomial$ with cell probabilities $\frac{1}{2} + \frac{\theta}{4}, \frac{1}{4}(1-\theta), \frac{1}{4}(1-\theta), \frac{\theta}{4})$. Find a sufficient and minimal sufficient statistic for $\theta$.

10. Let $N$ be a random variable taking values $1, 2, ...$ with known probabilities $p_1, ...$ where $\sum_i p_i = 1$. Having observed $N = n$, perform $n$ Bernoulli trials with success probability $\theta$, getting $X$ successes.

    - Prove that the pair $(X, N)$ is minimal sufficient and $N$ is ancillary for $\theta$.

    - Prove that the estimator $\frac{X}{N}$ is unbiased for $\theta$ and has variance $\theta(1-\theta)E[\frac{1}{N}]$.

11. $X_1, ..., X_{2m+1} \sim N(\mu, \sigma^2)$, $\sigma^2$ known. Prove that $Var(X_{(m+1)}) \geq Var(\bar{X}_{2m+1})$.

12. Let $X_1, ..., X_n \overset{iid}{\sim} f\left(\frac{x-\mu}{\sigma}\right)$. Let $T_1(X_1, ..., X_n)$ and $T_2(X_1, ..., X_n)$ be two statistics that both satisfy $T_i(ax_1 + b, ..., ax_n + b) = aT_i(x_1, ..., x_n)$, for all values of $x_1, ..., x_n$ and $b$ and for any $a > 0$.

- Show that $T_1(\boldsymbol{X})/T_2(\boldsymbol{X})$ is an ancillary statistic.
- Let $R$ be the sample range $R = X_{(n)} - X_{(1)}$ and $S = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Then $R/S$ is an ancillary statistic.

13. Let $X$ takes on values $-1, 0, 1, 2, ...$ with probabilities

$$P(X = -1) = p, \;\; P(X = k) = (1-p)^2 p^k, \; k = 0, 1...$$

where $0 < p < 1$.

- Prove that $U(X)$ is an unbiased estimator of 0 if and only if $U(k) = -kU(-1)$ for all $k = 0, 1....$
- Let

$$\delta_0(X) = \begin{cases} 1 \text{ if } X = -1 \\ 0 \text{ o.w.} \end{cases}$$

$\delta_0$ is an unbiased estimator of 0. If $U(-1) \neq 0$, prove that $\delta_0$ cannot be the UMVUE.

14. $X_1, ..., X_n \overset{iid}{\sim} Bernoulli(p)$. Provide the UMVUE of $p(1-p)$.

15. $X_1, ..., X_n \overset{iid}{\sim} N(\mu, 1)$. Provide UMVUE of $P(X_1 \leq u) = \Phi(u - \mu)$ for some $u$.

16. $X_1, ..., X_n \overset{iid}{\sim} f_{\theta_1,\theta_2}(x)$ where

$$f_{\theta_1,\theta_2}(x) = \frac{1}{\theta_2}\exp(-(x-\theta_1)/\theta_2), \;\; x > \theta_1.$$

- Show that $X_{(1)}$ and $\sum_{i=1}^{n}[X_i - X_{(1)}]$ are jointly sufficient and complete.
- Show that $X_{(1)}$ and $\sum_{i=1}^{n}[X_i - X_{(1)}]$ are independent.
- Find UMVUE for $\theta_1$ and $\theta_2$.

17. $X_1, ..., X_n \overset{iid}{\sim} U(0, \theta)$, $Y_1, ..., Y_m \overset{iid}{\sim} U(0, \theta_1)$ and all $X$ and $Y$ are independent. Find UMVUE for $\theta/\theta'$.

18. $X_1, ..., X_n \overset{iid}{\sim} Pois(\lambda)$. Prove that $E[S^2|\bar{X}] = \bar{X}$, hence $var(S^2) < var(\bar{X})$.

19. Suppose $X$ and $Y$ are independently distributed with densities $f_\theta$ and $q_\theta$ respectively. If $I_1(\theta), I_2(\theta)$ and $I(\theta)$ are the information of $X$, $Y$ and $(X, Y)$ respectively, prove that $I(\theta) = I_1(\theta) + I_2(\theta)$.

20. For each of the following distributions, let $X_1, ..., X_n$ be a random sample. Is there a function of $\theta$, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramer-Rao lower bound? If so, find it.

    - $f_\theta(x) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$.
    - $f_\theta(x) = \frac{\log(\theta)}{\theta-1}\theta^x$, $0 < x < 1$, $\theta > 1$.

21. Let $X_1, ..., X_n$ be a random sample from a population with p.m.f $P_\theta(X = x) = \theta^x(1 - \theta)^{1-x}$, $x = 0, 1$, $0 \le \theta \le 1/2$.

    - Find the method of moment estimator and MLE of $\theta$.
    - Find the mean squared errors for each of them.
    - Which one do you prefer based on mean squared errors?

22. Let $X_1, ..., X_{2m+1}$ follows iid $f_\theta(x) = \frac{1}{2}\exp(-|x - \theta|)$, $-\infty < x < \infty$. Find the MLE of $\theta$.

23. The LINEX loss is given by

$$L(\theta, a) = e^{c(a-\theta)} - c(a - \theta) - 1,$$

where $c$ is a positive constant. Show that the Bayes estimator of $\theta$ under this loss function, using a prior $\pi$ is $\delta^\pi(X) = \frac{-1}{c}\log[E(e^{-c\theta}|X)]$. Evaluate this estimator when

$X_1, ..., X_n \sim N(\theta, 1)$ and $\pi(\theta) = 1$.