Assignment 6

March 4, 2017

1. Let $X_1, ..., X_n \sim N(0, \sigma_X^2)$ and $Y_1, ..., Y_m \sim N(0, \sigma_Y^2)$, independent of X's. Define $\lambda = \sigma_Y^2/\sigma_X^2$.

   - Find a level α LRT of $H_0 : \lambda = \lambda_0$ vs. $H_1 : \lambda \neq \lambda_0$.
   - Express the rejection region of LRT in terms of F random variable.
   - Find a $1 - \alpha$ confidence interval for λ.

2. Let $X_1, ..., X_n$ be independent with p.d.f $f_{X_i}(x) = e^{i\theta - x} I_{[i\theta,\infty)}(x)$. Based on $T = min_i(X_i/i)$, find the $1 - \alpha$ confidence interval of θ of the form $[T + a, T + b]$ which is of minimum length.

3. Let $X_1, ..., X_n$ be iid $U(0, \theta)$. Let Y be the largest order statistic. Prove that Y/θ is the pivotal quantity and show that the interval $\{\theta : y \leq \theta \leq y/\alpha^{1/n}\}$ is the shortest $1 - \alpha$ pivotal quantity.

4. $X_1, ..., X_n$ be iid exponential(λ)

   - Find a UMP size α hypothesis test of $H_0 : \lambda = \lambda_0$ vs. $H_1 : \lambda < \lambda_0$.
   - Find a UMA $1 - \alpha$ confidence interval based on inverting the above test. Show that the interval can be expressed as

$$C^*(X_1, ..., X_n) = \{\lambda : 0 \leq \lambda \leq \frac{2\sum_{i=1}^n X_i}{\chi^2_{2n,\alpha}}\}.$$

- Find the expected length of $C^*(X_1, ..., X_n)$.

5. Show that if $A(\theta_0)$ is an unbiased level α acceptance region of a test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ and $C(\boldsymbol{X})$ is the $1 - \alpha$ confidence set formed by inverting the acceptance regions, then $C(\boldsymbol{X})$ is an unbiased $1 - \alpha$ confidence set.

6. Let $X \sim Beta(\theta, 1)$.

   - Let $Y = -(\log X)^{-1}$. Evaluate the confidence coefficient of the set $[Y/2, Y]$.
   - Find a pivotal quantity and use it to set up a confidence interval having the same confidence coefficient.
   - Compare the two confidence intervals.

7. $X_1, ..., X_n$ be iid observations from an exponential(λ) pdf, λ has conjugate prior $IG(a, b)$. Show how to find a $1 - \alpha$ Bayes HPD credible set for λ.

8. Let $X \sim f(x)$ where f is a strictly decreasing pdf on $[0, \infty)$. For a fixed value of $1 - \alpha$, of all intervals $[a, b]$ that satisfy $\int_a^b f(x)dx = 1 - \alpha$, the shortest is obtained by $a = 0$ and b s.t. $\int_0^b f(x)dx = 1 - \alpha$.

9. Let $X_1, ..., X_n$ be a random sample from $N(\theta, \sigma^2)$.

   - Show that the interval $\theta \leq \bar{X} + t_{n-1,\alpha} \frac{s}{\sqrt{n}}$ can be derived by inverting the acceptance region of an LRT.
   - Show that the interval is an unbiased interval.