Recap: Most powerful test for testing simple null vs. simple alternative hypothesis.

$H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$

- If $f(\underline{x}|\theta_i)$ $i = 0, 1$ is the pmf or pdf of $\underline{x}$ under null and alternative hypotheses respectively, then the MP test of level α is given by

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \dfrac{f(\underline{x}|\theta_1)}{f(\underline{x}|\theta_0)} > K \\ 0 & \text{o.w.} \end{cases}$$

where $\alpha = P_{\theta_0}(\underline{X} \in \mathcal{R})$.

$\mathcal{R}$ = rejection region.

By factorization theorem,

$$f(\underline{x}|\theta_1) = g(T(\underline{x}), \theta_1)\, \cancel{h(\underline{x})}$$

and $$f(\underline{x}|\theta_0) = g(T(\underline{x}), \theta_0)\, \cancel{h(\underline{x})}$$

$$\Rightarrow \frac{f(\underline{x}|\theta_1)}{f(\underline{x}|\theta_0)} = \frac{g(T(\underline{x}), \theta_1)}{g(T(\underline{x}), \theta_0)}$$

$$\phi(t) = \begin{cases} 1 & \text{if } g(t, \theta_1) > K\, g(t, \theta_0) \\ 0 & \text{o.w.} \end{cases}$$

Where T is a sufficient statistic.

Example: $X_1, X_2 \overset{iid}{\sim} Ber(\theta)$

we want to test $H_0: \theta = \frac{1}{2}$ vs. $H_1: \theta = \frac{3}{4}$.

Sufficient statistic $\sum_{i=1}^{2} X_i \sim Bin(2, \theta)$

$$\frac{f(t=0 \mid \theta = \frac{3}{4})}{f(t=0 \mid \theta = \frac{1}{2})} = \frac{1}{4}, \quad \frac{f(t=1 \mid \theta = \frac{3}{4})}{f(t=0 \mid \theta = \frac{1}{2})} = \frac{3}{4},$$

$$\frac{f(t=2 \mid \theta = \frac{3}{4})}{f(t=2 \mid \theta = \frac{1}{2})} = \frac{9}{4}.$$

Choose $\frac{3}{4} < k < \frac{9}{4}$. $\Rightarrow R = \{2\}$.

$$P_{\theta=\frac{1}{2}}(T=2) = \alpha = \frac{1}{4}.$$

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \frac{f(t \mid \theta = \frac{3}{4})}{f(t \mid \theta = \frac{1}{2})} > k \\ 0 & \text{o.w.} \end{cases}$$

for $\frac{3}{4} < k < \frac{9}{4}$.

Choose $\frac{1}{4} < k < \frac{3}{4}$. $\Rightarrow R = \{1, 2\}$.

$$P_{\theta=\frac{1}{2}}(T=1) + P_{\theta=\frac{1}{2}}(T=2) = \frac{3}{4}$$

Therefore $\phi(\underline{x}) = \begin{cases} 1 & \text{if } \frac{f(t \mid \theta = \frac{3}{4})}{f(t \mid \theta = \frac{1}{2})} > k \\ 0 & \text{o.w.} \end{cases}$

for $\frac{1}{4} < k < \frac{3}{4}$, is the MP test of level $\frac{3}{4}$.

If chose $k < \frac{1}{4} \Rightarrow R = \{0, 1, 2\}$

So, it is the MP test of level 1.

Similarly choosing $K > \frac{9}{4}$ gives us the MP test of level 0.

<u>Example</u>: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known.
We would like to test $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$.
Sufficient statistic for μ is $\bar{X}$.

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

$$\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \dfrac{g(\bar{x}|\mu_1, \sigma^2)}{g(\bar{x}|\mu_0, \sigma^2)} > K \\ 0 & \text{o.w.} \end{cases}$$

where, $P_{\mu=\mu_0}\left(\dfrac{g(\bar{x}|\mu_1, \sigma^2)}{g(\bar{x}|\mu_0, \sigma^2)} > K \right) = \alpha$.

$$\frac{g(\bar{x}|\mu_1, \sigma^2)}{g(\bar{x}|\mu_0, \sigma^2)} > K \iff \bar{x} < \frac{(2\sigma^2 \log K)/n - \mu_0^2 + \mu_1^2}{2(\mu_1 - \mu_0)}$$

Thus K is determined by the equation

$$\underset{\mu=\mu_0}{P}\left(\bar{X} < \frac{(2\sigma^2 \log K)/n - \mu_0^2 + \mu_1^2}{2(\mu_1 - \mu_0)} \right) = \alpha$$

$$\iff \underset{\mu=\mu_0}{P}\left(\underbrace{\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}}_{N(0,1)} < \frac{\frac{(2\sigma^2 \log K)}{n} - \mu_0^2 + \mu_1^2 - \mu_0}{2(\mu_1 - \mu_0)\sigma/\sqrt{n}} \right) = \alpha$$

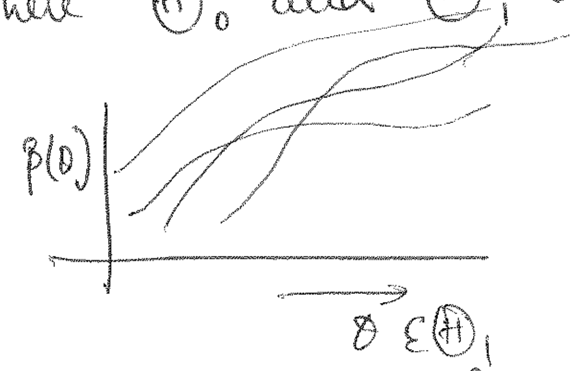
$$\Rightarrow z_{1-\alpha} = \frac{\frac{2\sigma^2 \log K}{n} - \mu_0^2 + \mu_1^2 - \mu_0}{2(\mu_1 - \mu_0)\sigma/\sqrt{n}}.$$

Solve for K.

Neyman - Pearson lemma provides the most powerful test for a simple null vs. simple alternative hypothesis. How to construct tests ~~that~~ which generalize this idea to testing

$H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1$.

where Θ_0 and Θ_1 are not singleton sets.



Here we will define uniformly most powerful test.

<u>Definition</u>: Let C be a class of tests for testing $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_0^c$. A test in class C, with a power function $\beta(\theta)$ is the uniformly most powerful (UMP) class C test if $\beta(\theta) \geq \beta_1(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta_1(\theta)$ that is a power function of a test in class C.

UMP test can't be found for a very general Θ_0. There are some cases where it can be found. We will study one such case.

Monotone likelihood ratio : The family of distribution $\{f(\underline{x}|\theta) : \theta\}$ is said to have monotone likelihood ratio in $T(\underline{x})$ if $\frac{f(\underline{x}|\theta_1)}{f(\underline{x}|\theta_0)}$ is an nondecreasing function of $T(\underline{x})$ for all $\theta_1 \geq \theta_0$.

Example: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\frac{f(\underline{x}|\mu_1)}{f(\underline{x}|\mu_0)} = \frac{\cancel{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n} \exp\left\{-\sum_{i=1}^{n} \frac{(x_i-\mu_1)^2}{2\sigma^2}\right\}}{\cancel{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n} \exp\left\{-\sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\sigma^2}\right\}}$$

$$= \exp\left\{\sum_{i=1}^{n} x_i \left(\frac{\mu_1-\mu_0}{\sigma^2}\right) - \frac{n}{2\sigma^2}(\mu_1^2-\mu_0^2)\right\}$$

if $\mu_1 > \mu_0$ then $\frac{f(\underline{x}|\mu_1)}{f(\underline{x}|\mu_0)}$ is a nondecreasing function of $\sum_{i=1}^{n} x_i$.

$\Rightarrow$ the family of distribution $\{N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 \text{ is known}\}$ has an MLR in $\sum_{i=1}^{n} x_i$.

We will discuss UMP when the family of distributions has an MLR.

$T(x) > T(x_1)$ $\quad \frac{f(x|\theta_1)}{f(x|\theta_0)}$ nondecreasing fn.

$\underline{g(T(\underline{x}))} < g(T(x_1))$ $\quad \frac{f(x|\theta_1)}{f(x|\theta_0)}$ nondecreasing fn.

back ①

Suppose ~~$T(x)$~~ parametric family $f(x|\theta)$ has MLR in $T(\underset{\sim}{x})$. Let $\theta_1 > \theta_0$, we already know by NP lemma that

$$\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \dfrac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > k \\ 0 & \text{o.w.} \end{cases} \qquad \alpha = P\left(\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > k\right)$$

is the MP test of certain level dependent on k.

$$\left\{\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > k\right\} \Leftrightarrow \{T(\underset{\sim}{x}) > b\}$$

$$\left.\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } T(\underset{\sim}{x}) > b \\ 0 & \text{o.w.} \end{cases}\right\} \quad (1)$$

$P_{\theta_0}(T(\underset{\sim}{X}) > b) = \alpha$.

$H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$ for $\theta_1 > \theta_0$

if we look at the most powerful test for any $\theta_1 > \theta_0$, it is the same test given by (1).

The above test (1) is therefore UMP test for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$.

back (2)

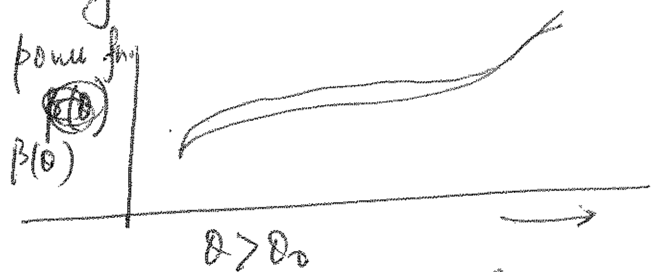
test ① is MP for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$.

$\theta_1 > \theta_0$.

- this is MP test for testing lets say $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_0 + 5$

also this MP test for $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_0 + 20$

take any other test.



thus ① is UMP for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$.

Remark: It can be shown that the power function $\beta(\theta) = E_\theta(\phi)$, is non-decreasing for this test.

Pf: $\beta(\theta) = P_\theta(T > b)$

I have to show if $\theta_1 > \theta_2$ then $\beta(\theta_1) \geq \beta(\theta_2)$.

$$\frac{d}{db}\left[P_{\theta_1}(T \leq b) - P_{\theta_2}(T \leq b)\right] = f_{\theta_1}(b) - f_{\theta_2}(b)$$

$$= f_{\theta_2}(b)\left(\frac{f_{\theta_1}(b)}{f_{\theta_2}(b)} - 1\right)$$

~~this ratio~~ since $\frac{f_{\theta_1}(b)}{f_{\theta_2}(b)}$ is non-decreasing

the above quantity, if changes sign, can only change from negative to positive.

back ③

If it indeed changes sign from negative to positive it means that the function ~~at~~ reaches a local minimum at that point. So, the function is maximized either at $-\infty$ or at $+\infty$. At both these points the fn.

$P_{\theta_1}(T \le b) - P_{\theta_2}(T \le b)$ is 0.

$\Rightarrow P_{\theta_1}(T \le b) - P_{\theta_2}(T \le b) \le 0$.

$\Rightarrow P_{\theta_1}(T > b) \geq P_{\theta_2}(T > b)$

$\Rightarrow$ this function is a non-decreasing function.

Since the power function is nondecreasing and $\beta(\theta_0) = \alpha$

$\Rightarrow \beta(\theta) \le \alpha \quad \forall \; \theta \le \theta_0$.

$\Rightarrow$ If I take any test for which the power function $\beta_1(\theta) \le \alpha \quad \forall \; \theta \le \theta_0$ then test (1) is uniformly most powerful among all such tests.

$\Rightarrow H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$ test (1) is UMP among all tests of level α.

back (4)

consider ~~the~~ finding the UMP test for

$H_0: \theta \geq \theta_0$ vs. $H_1: \theta < \theta_0$.

Result: $\phi(\underline{X}) = \begin{cases} 1 & \text{if } T(\underline{x}) < b \\ 0 & \text{if } \cancel{T(\underline{x}) \geq b} \text{ o.w.} \end{cases}$

s.t. $P_{\theta_0}(T(\underline{X}) < b) = \alpha$.

Pf:- Start with the simple null hypothesis

$H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$, $\theta_1 < \theta_0$

Since $f_\theta(\cdot)$ has MLR in $T(\underline{X})$,

$\Rightarrow \dfrac{f_{\theta_0}(\underline{X})}{f_{\theta_1}(\underline{X})}$ is a non-decreasing fn. of $T(\underline{X})$

$\Rightarrow \dfrac{f_{\theta_1}(\underline{X})}{f_{\theta_0}(\underline{X})}$ is a non-decreasing fn. of $-T(\underline{X})$

From here we ~~will be able~~ can conclude similar to the earlier proof that

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } -T(\underline{X}) > b \\ 0 & \text{o.w.} \end{cases}$$

$$\cancel{\phi(\underline{X}) = \begin{cases} 1 & \text{if } T(\underline{X}) < -b = b_1 \\ 0 & \text{o.w.} \end{cases}}$$

$$\Leftrightarrow \phi(\underline{X}) = \begin{cases} 1 & \text{if } T(\underline{X}) < -b = b_1 \\ 0 & \text{o.w.} \end{cases}$$

Remark: One parameter exponential family has an MLR in $\sum_{i=1}^{n} T(x_i)$.

$$\exp\left\{ w(\theta)\, T(x) \right\} h(\theta)$$