Recap:

We talked about finding the UMP test for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$, when the family of distributions $\{f(\cdot|\theta) : \theta \in \mathbb{R}\}$ assumes an MLR.

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } T(\underline{x}) > b \\ 0 & \text{o.w.} \end{cases}$$

$P_{\theta=\theta_0}(T(\underline{x}) > b) = \alpha$ is the UMP test of level α.

Example: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ is known.

$H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$

We have seen that the $\{N(\theta, \sigma^2) : \theta \in \mathbb{R}\}$ has MLR in $\sum_{i=1}^{n} X_i$. Thus UMP test

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > b \\ 0 & \text{o.w.} \end{cases}$$

$$P_{\theta_0}\left(\sum_{i=1}^{n} X_i > b\right) = \alpha$$

$$\Leftrightarrow \phi(\underline{X}) = \begin{cases} 1 & \text{if } \sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) > c \\ 0 & \text{o.w.} \end{cases}$$

$$P_{\theta_0}\left(\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) > c\right) = \alpha$$

We know $\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) \sim N(0,1)$ under $\theta = \theta_0$

$$P_{\theta_0}\left(\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) > c\right) = \alpha \Leftrightarrow P(N(0,1) > c) = \alpha$$

thus $c = z_\alpha$ which can be found from the normal table the UMP test is

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } \sqrt{n}\,\dfrac{(\bar{X}-\theta_0)}{\sigma} > z_\alpha \\ 0 & \text{o.w.} \end{cases}$$

So, the classical one-sided Normal test is the UMP test.

<u>p-value</u>: ~~In many~~ In NP lemma $K = K(\alpha)$ is determined by using the fact that the prob. of $\dfrac{f_{\theta_1}(\underline{X})}{f_{\theta_0}(\underline{X})} > K$ is α. Let the rejection region for level α be denoted by R_α. In many situations the region R_α is nested in a way that $R_\alpha \subset R_{\alpha'}$ if $\alpha < \alpha'$. When this is the case, it is a common practice to see not only whether the hypothesis is accepted or rejected, but also determine the smallest level at which the hypothesis is rejected. This is called the p-value. Mathematically

$$p(\underline{X}) = \inf\{\alpha : \underline{X} \in R_\alpha\}.$$

<u>Example</u>: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$

$H_0: \mu = 0$ vs. $H_1: \mu > 0$. What is the rejection region of ~~a~~ the UMP test of level α.

$$R_\alpha = \left\{\bar{X} : \bar{X} > \frac{1}{\sqrt{n}} z_{1-\alpha}\right\} = \left\{\bar{X} : 1 - \Phi(\sqrt{n}\,\bar{X}) < \alpha\right\}$$

Given data $\inf\{\alpha : \underline{X} \in R_\alpha\} = 1 - \Phi(\sqrt{n}\,\bar{X})$

Remark: If we are interested in the one parameter exponential family with parameter θ, does UMP exist for testing

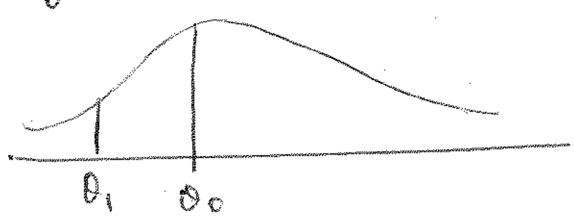
$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$.

Answer: NO

Let us look at the result in the case of Normal distribution.

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known. For any $\theta_1 < \theta_0$ the UMP level α-test for testing

$H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$



$H_0: \theta \geq \theta_0$ vs. $H_1: \theta < \theta_0$

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } T(\underline{X}) < b \\ 0 & \text{o.w.} \end{cases}$$

$$\Leftrightarrow \quad \phi(\underline{X}) = \begin{cases} 1 & \text{if } \sqrt{n}\left(\frac{\bar{X} - \theta_0}{\sigma}\right) < b \\ 0 & \text{o.w.} \end{cases}$$

$$P_{\theta_0}\Big(\underbrace{\sqrt{n}\left(\frac{\bar{X} - \theta_0}{\sigma}\right)}_{N(0,1)} < b\Big) = \alpha \qquad b = -z_\alpha.$$

the test is going to reject the null hypothesis if $\sqrt{n}\left(\frac{\bar{X} - \theta_0}{\sigma}\right) < -z_\alpha \Leftrightarrow \bar{X} < -\frac{z_\alpha \sigma}{\sqrt{n}} + \theta_0$

the UMP level α-test for testing

$H_0: \theta=\theta_0$ vs. $H_1: \theta=\theta_1$, $\theta_1<\theta_0$ is given

by $$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} < -z_\alpha \\ 0 & \text{o.w.} \end{cases}$$

~~power~~ calculate power fn. of this test at $\theta_2>\theta_0$

$$P_{\theta_2}\left(\sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} < -z_\alpha\right)$$

$$= P_{\theta_2}\left(\bar{X} < \theta_0 - \frac{z_\alpha \sigma}{\sqrt{n}}\right)$$

$$= P_{\theta_2}\left(\sqrt{n}\frac{(\bar{X}-\theta_2)}{\sigma} < \frac{\theta_0-\theta_2}{\sigma/\sqrt{n}} - z_\alpha\right)$$

$$< P(Z < -z_\alpha) \quad \text{where } Z \sim N(0,1)$$

this is because under θ_2, $\sqrt{n}\frac{(\bar{X}-\theta_2)}{\sigma} \sim N(0,1)$.

$$= P(Z > z_\alpha)$$

$$= P_{\theta_2}\left(\underbrace{\sqrt{n}\frac{(\bar{X}-\theta_2)}{\sigma}}_{N(0,1)} > z_\alpha\right)$$

$$= P_{\theta_2}\left(\bar{X} > \frac{\sigma z_\alpha}{\sqrt{n}} + \theta_2\right)$$

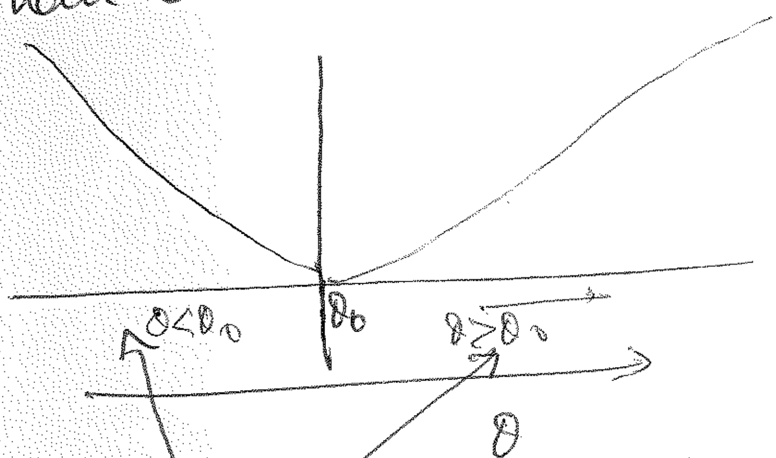
$$= P_{\theta_2}\left(\sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} > z_\alpha + \frac{(\theta_2-\theta_0)\sqrt{n}}{\sigma}\right)$$

$$< P_{\theta_2}\left(\sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} > z_\alpha\right) \quad \text{as } \theta_2 > \theta_0 .$$

when $\theta_2 > \theta_0$

$$P_{\theta_2}\left(\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} < -z_\alpha \right) < P_{\theta_2}\left(\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} > z_\alpha \right)$$

$\Rightarrow$ UMP test for testing $H_0: \theta=\theta_0$ vs. $H_1: \theta=\theta_1$ when $\theta_1<\theta_0$ is $\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} < -z_\alpha$ which has less power ~~that another test that rejects~~ ~~will~~ to test $H_0: \theta=\theta_0$ vs. $H_1: \theta=\theta_2$, $\theta_2>\theta_0$ than another test.

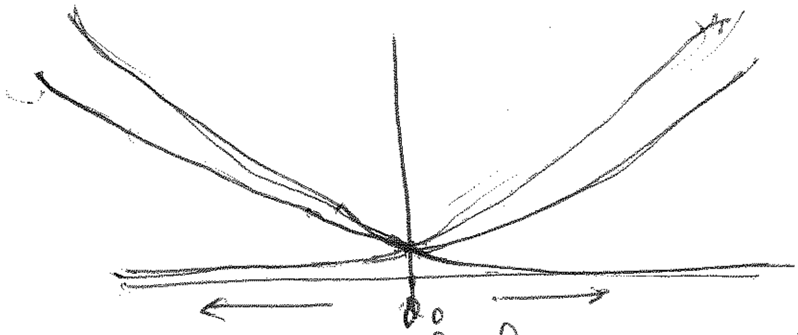


$\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} > z_\alpha$ is most powerful

$\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} < -z_\alpha$ is most powerful

$\Rightarrow$ $H_0: \theta=\theta_0$ vs. $H_1: \theta\neq\theta_0$ does not admit the UMP test.

$H_0: \theta=\theta_0$ vs. $H_1: \theta\neq\theta_0$ for $X_1, \ldots, X_n \overset{iid}{\sim} U(0,\theta)$, then $\exists$ UMP test.



the tests for which power function doesn't go below the level in the alternative are known as unbiased tests.

A test ϕ is unbiased at level α for testing $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1$ if

(i) $\beta(\theta) \leq \alpha$ for $\theta \in \Theta_0$

(ii) $\beta(\theta) \geqslant \alpha$ for $\theta \in \Theta_1$.

We want to find a test ϕ of level α, which is Most Powerful among all unbiased tests. If such a test exists then it is called UMPU test.

clearly, if the power function $\beta(\theta)$ of an unbiased test is cont. then in the boundary of Θ_0 and Θ_1, $\beta(\theta) = \alpha$

e.g., $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

$\beta(\theta_0) = \alpha$.

<u>Result</u>: If the densities are s.t. all tests have cont. power functions, then if $\exists$ a UMP test T among the tests satisfying $E[T'] = \alpha$ whenever θ belongs to the boundary of Θ_0 and Θ_1, then T is UMPU level α.

~~Application~~ UMPU tests in one parameter exponential family

$$p_\theta(x) = c(\theta)\, h(x) \exp(w(\theta) T(x))$$

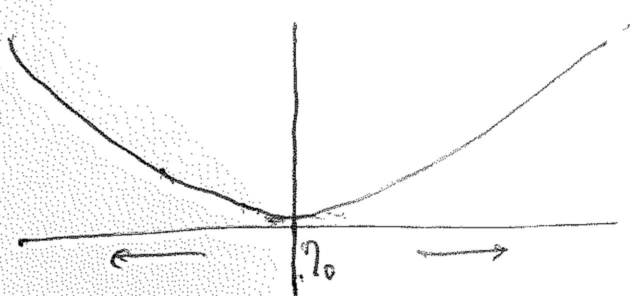
let $\eta = w(\theta) \Rightarrow p_\eta(x) = c_1(\eta)\, h(x) \exp(\eta T(x))$

This is called natural exponential family parametrization.

<u>UMPU test</u>: To test $H_0: \eta = \eta_0$ vs. $H_1: \eta \neq \eta_0$ the UMPU test is given by

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } T(\underline{x}) < c_1 \text{ or } T(\underline{x}) > c_2 \\ 0 & \text{o.w.} \end{cases}$$

$$\underline{E_{\eta_0}(\phi(\underline{x})) = \alpha}, \quad \underline{E_{\eta_0}(T(\underline{x})\, \phi(\underline{x})) = \alpha\, E_{\eta_0}(T(\underline{x}))}$$



$\beta(\eta) = E_\eta(\phi(\underline{x}))$
this should satisfy
$\underline{\beta'(\eta_0) = 0}$

If $T(\underline{X})$ follows a distribution symmetric around ~~under~~ k under H_0, then the above two conditions boil down to

$$E_{\eta_0}[\phi(\underline{X})] = \alpha, \quad c_1 + c_2 = 2k.$$

<u>Example</u>: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known.

To test. $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$.

Here normal ~~dist~~ belongs to the exponential family $\eta = \theta$ and $T(\underline{X}) = \bar{X}$.

UMPU test should look like

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } \bar{X} < c_1 \text{ or } \bar{X} > c_2 \\ 0 & \text{o.w.} \end{cases}$$

$$\Leftrightarrow \phi(\underline{X}) = \begin{cases} 1 & \text{if } \sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} < b_1 \text{ or } \sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} > b_2 \\ 0 & \text{o.w.} \end{cases}$$

Subject. $E_{\theta_0}[\phi(\underline{X})] = \alpha$ and $E_{\theta_0}\left[\phi(\underline{X})\sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma}\right] = \alpha E_{\theta_0}\left[\sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma}\right]$

$T(\underline{X}) = \sqrt{n}\frac{(\bar{X}-\theta_0)}{\sigma} \sim N(0,1)$ under H_0.

$\Rightarrow T(\underline{X})$ has symmetric dist. around 0 under H_0.

$$E_{\theta_0}\left(\phi(\underline{X})\right) = \alpha \quad \cdots\cdots (1)$$

$$b_1 + b_2 = 2.0 = 0 \quad \cdots\cdots (2)$$

$$\int_{-\infty}^{b_1} f(z)\,dz + \int_{b_2}^{\infty} f(z)\,dz = \alpha$$ when f is the density of $N(0,1)$.

$E_{\theta_0}\left[\phi(\underline{X})\right] = \alpha$

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } \sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) < b_1 \text{ or } \sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) > b_2 \\ 0 & \text{o.w.} \end{cases}$$

$$E_{\theta_0}\left[\phi(\underline{X})\right] = P_{\theta_0}\left(\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) < b_1\right) + P_{\theta_0}\left(\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right) > b_2\right)$$

$$= \Phi(b_1) + 1 - \Phi(b_2)$$

thus ① & ② become

$b_1 + b_2 = 0$ . . . . . ③

$\Phi(b_1) + 1 - \Phi(b_2) = \alpha$ . . . ④

$\Rightarrow b_1 = -b_2$

$\Rightarrow \Phi(-b_2) + 1 - \Phi(b_2) = \alpha$

$\Rightarrow 2\Phi(-b_2) = \alpha \Rightarrow \Phi(-b_2) = \frac{\alpha}{2}$

$\Rightarrow b_2 = z_{1-\alpha/2}$ $\quad b_1 = -b_2 = z_{\alpha/2} = -z_{1-\alpha/2}$

$$\phi(\underline{X}) = \begin{cases} 1 & \text{if } \left|\sqrt{n}\left(\frac{\bar{X}-\theta_0}{\sigma}\right)\right| > z_{1-\alpha/2} \\ 0 & \text{o.w.} \end{cases}$$

the classical two sided test is the UMPU test of level α.

back ①

Likelihood ratio test

The aim is to test $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_1$.
$= \Theta \setminus \Theta_0$

Based on samples $X_1, \ldots, X_n$.
Let $f_\theta(\underline{X})$ be the likelihood of θ. The likelihood ratio test (LRT) ~~is given~~ statistic is given by

$$\lambda = \frac{\sup_{\theta \in \Theta_0} f_\theta(\underline{X})}{\sup_{\theta \in \Theta} f_\theta(\underline{X})}$$

If Θ_0 contains not very probable θ values then $\sup_{\theta \in \Theta_0} f_\theta(\underline{X})$ should be much smaller than $\sup_{\theta \in \Theta} f_\theta(\underline{X})$

Thus H_0 is rejected when $\lambda < c$, when c is determined by the level of the test.
i.e. $\sup_{\theta \in \Theta_0} P_\theta(\lambda < c) \leq \alpha$. Since $0 < \lambda < 1$
so is c.

back ②